JOSE C. PRINCIPE
U. OF FLORIDA
EEL 6502
352-392-2662
principe@cnel.ufl.edu

TRACKING PERFORMANCE OF ADAPTIVE FILTERS

Nonstationary data model

Let me assume that for any given data {d(i),x(i)}

$$d(i) = x(i)\, w_i^* + v(i)$$

where v(i) is the unmodeled component of the desired and is orthogonal to x(i) with variance $J_{min}(i) = \sigma_v^2(i)$. We will further assume that it is i.i.d. and independent to all {x(i)}.

We will also assume a model for the variation of the weight vectors $w^*(i) = w^*(i-1) + q(i)$

where q(i) is a random perturbation independent of {x(j),v(j)} assumed zero mean with covariance

$$Q = E\left[q(i)\, q^T(i) \right]$$

It is easy to see that the weights will have constant mean. The initial condition is also a random variable independent

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It is easy to see that the weights at each time j are dependent of $W(j) = f\left[W_{-1}; d(j) \dots d(0); x(j), \dots x(0)\right]$

Now v(i) is independent of w(j) for j<i. v(i) is also independent of $\tilde{w}$(j) defined as $\tilde{W}(j) = w^*(j) - w(j)$
and also of the a priori error defined in the nonstationary case as $e_a(i) = x(i)\,w^*(i) - x(i)\,w(i-1)$ $\left[\underline{\text{NOTE:}}\ e_a(i) \neq x(i)\,w^*(i-1)\right]$

Excess MSE (EMSE) in the nonstationary case

Write
$$e(i) = d(i) - x(i)\,w(i-1)$$
$$= v(i) + e_a(i)$$

Since v(i) is zero mean and ea(i) are independent
$$E\left[e(i)\right|^2 = E\left|v(i)\right|^2 + E\left|e_a(i)\right|^2$$
$$\Longrightarrow \text{EMSE} = \lim_{i \to \infty} E\left|e_a(i)\right|^2$$

So the EMSE in the nonstationary case can be ALSO evaluated by the steady state variance of the a priori error.

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A lower bound to the EMSE can be determined as

$$e_a(i) = v(i)\,w^*(i) - v(i)\,w(i-1) = x(i)\left[w^*(i-1) - w(i-1)\right] + x(i)q(i)$$

$$E|e_a(i)|^2 = E|x(i)(w^*(i-1) - w(i-1)|^2 + E|x(i)q(i)|^2$$

$$\geq E|x(i)q(i)|^2 = \mathrm{Tr}(RQ) \qquad \forall i$$

So the misadjustment of an adaptive filter in a nonstationary environment is lowered bounded by

$$M \geq \frac{\mathrm{Tr}(RQ)}{\sigma_v^2}$$

The degree of stationarity of the data is defined as

$$DN \equiv \sqrt{\frac{\mathrm{Tr}(RQ)}{\sigma_v^2}}$$

If DN is larger than one, then the filter will not be able to track them and misadjustment will be large. In the other cases the filter can track.

JOSE C. PRINCIPE
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Fundamental Energy Conservation Relations

Let us consider any adaptive filter obeying to the condition

$$w(i) = w(i-1) + \eta\, x(i)\, g(e(i)) \qquad w(-1) \rightarrow \text{INITIAL COND.}$$

which can be written

$$w^*(i) - w(i) = (w^*(i) - w(i-1)) - \eta\, x(i)\, g(e(i))$$

Multiplying from the left both sides by x(i) the a priori and a posteriori errors are related by

$$e_p(i) = e_a(i) - \eta \|x(i)\|^2 g(e(i))$$

Therefore we have (energy conservation relation)

$$\|w^*(i) - w(i)\|^2 + \eta(i)|e_a(i)|^2 = \|w^*(i) - w(i-1)\| + \eta(i)|e_p(i)|^2$$

$$\begin{cases} e_a(i) = x(i)(w^*(i) - w(i-1)) \\ e_p(i) = x(i)(w^*(i) - w(i)) \end{cases}$$

$$\eta(i) = \frac{1}{\|x(i)\|^2} \quad x(i) \neq 0 \qquad (0 \text{ for } x(i)=0)$$

Note: In the case of steady state behavior we just need to re-interpret the weights.

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Fundamental Variance Relation

From this we can get the variance $\left(w^*(i) = w^*(i-1) + q(i) \right)$

$$E\|\tilde{w}(i)\|^2 + E\left(\eta(i)|e_a(i)|^2\right) = E\|w^*(i) - w(i-1)\|^2 + E\left(\eta(i)|e_p(i)|^2\right)$$

And from the random walk model we have

$$E\|w^*(i) - w(i-1)\|^2 = E\|\tilde{w}(i-1) + q(i)\|^2$$

$$= E\|\tilde{w}(i-1)\|^2 + E\left[\|q(i)\|^2\right] + \cancelto{0}{E\left[\tilde{w}^T(i-1)q(i)\right]} + \cancelto{0}{E\left[q^T(i)\tilde{w}(i-1)\right]}$$

We can show that the crossterms are zero due to independence so

$$E\|\tilde{w}(i)\|^2 + E\left(\eta(i)|e_a(i)|^2\right) = E\|\tilde{w}(i-1)\|^2 + Tr(Q) + E\left[\eta(i)|e_p(i)|^2\right)$$

For steady state performance $E\|\tilde{w}(i)\|^2 = E\|\hat{w}(i-1)\|^2$

$i \rightarrow \infty$

$$E\left[\eta(i)|e_a(i)|^2\right] = TR(Q) + E\left[\eta(i)\left|e_a(i) - \eta\|x(i)\|^2 g(e(i))\right|^2\right]$$

$$\eta\, E\left[\|x(i)\|^2 |g(e(i))|^2\right] + \eta^{-1} Tr(Q) = 2E\left[e_a(i)\, g[e(i)]\right]$$

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Tracking Performance of the LMS

In the case of LMS

$$w(i) = w(i-1) + \eta\, x(i)\, e(i)$$

$$g[e(i)] = e(i) = e_a(i) + v(i)$$

$$\eta\, E\left[\|x(i)\|^2 |e_a(i) + v(i)|^2 + \eta^{-1} Tr(Q) = 2 E\left(e_a(i)(e_a(i) + v(i))\right)\right]$$

For small stepsizes we get

$$E(|e_a(\infty)|^2) = \frac{1}{2}\left[\eta\, E\left[\|x(i)\|^2 |e_a(i)|^2 + \eta \sigma^2 Tr(R) + \eta^{-1} Tr(Q)\right]\right]$$

$$E(|e_a(\infty)|^2) = \frac{\eta \sigma^2 Tr(R) + \eta^{-1} Tr(Q)}{2}$$

If we assume independence of inputs and the a priori error

$$\eta_{OPT} = \sqrt{\frac{Tr(Q)}{\sigma^2 T(R)}}$$

$$E(|e_a(\infty)|^2)_{min} = \sqrt{\sigma^2 Tr(R)\, Tr(Q)}$$